

References

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Eigenvector Derivatives of Repeated Eigenvalues Using Singular Value Decomposition

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Introduction

THE computational problem of eigenvector derivatives corresponding to repeated eigenvalues has not been treated adequately in the past, whereas several techniques are available for distinct eigenvalues.¹ The main difficulty appears to be the nonuniqueness of eigenvectors for repeated eigenvalues. For symmetric eigenvalue problems, a method² has been proposed for computing the eigenvector derivatives of repeated eigenvalues. A computational method has been proposed³ recently for general nondefective matrices, although the results are inconclusive.

In this Note, explicit formula for the first-order eigenvector derivative corresponding to the eigenvector of a repeated eigenvalue is derived for the non-self-adjoint eigenvalue problem. The method applies to the class of nondefective problems whose first eigenvalue derivatives of the repeated eigenvalues are distinct. A singular value decomposition (SVD)⁴ approach is used to compute four *required* bases for eigenspaces, and to keep track of the dimensions of state variables and the conditioning of the state equations.

Computation of Bases for Eigenspaces

Consider the n th order eigenvalue problem

$$(A - \lambda_i I)x_i = 0 \quad (1)$$

where A is assumed nondefective. Define the following notations

$$A_i \triangleq A - \lambda_i I, \quad r_i \triangleq \text{rank}(A_i), \quad \nu_i \triangleq n - r_i; \quad i = 1, \dots, N$$

where N is the number of distinct eigenvalues and ν_i the dimension of the null space of A_i or, equivalently, the number of linearly independent eigenvectors associated with λ_i . Thus, for a simple eigenvalue, $\nu_i = 1$.

Although any linear combination of the multiple eigenvectors is an eigenvector, let $\{x_i^{(1)}, x_i^{(2)}, \dots, x_i^{(\nu_i)}\}$ be some choice of right eigenvectors corresponding to λ_i . Consider the SVD of A_i

$$A_i = [\tilde{U}_i U_i] \begin{bmatrix} \Sigma_i & \\ & 0 \end{bmatrix} \begin{bmatrix} \tilde{V}_i^H \\ V_i^H \end{bmatrix} = \tilde{U}_i \Sigma_i \tilde{V}_i^H \quad (2)$$

where $\Sigma_i = \text{diag}[\sigma_1, \dots, \sigma_{r_i}]$, $[\tilde{U}_i U_i]$, and $[\tilde{V}_i V_i]$ denote the non-zero singular values, left and right singular vectors, respectively. The superscript H represents Hermitian transpose. The following relations hold:

$$U_i^H \tilde{U}_i = V_i^H \tilde{V}_i = 0, \quad \tilde{U}_i^H \tilde{U}_i = \tilde{V}_i^H \tilde{V}_i = I_{r_i}, \quad U_i^H U_i = V_i^H V_i = I_{\nu_i} \quad (3)$$

The $(n \times \nu_i)$ matrix V_i is an orthonormal basis for the right eigenvectors so that the k th eigenvector corresponding to the i th eigenvalue can be expressed as

$$x_i^{(k)} = V_i \alpha_i^{(k)}; \quad k = 1, \dots, \nu_i \quad (4)$$

$\alpha_i^{(k)}$ denotes the K th eigenvector coefficients (or coordinates in V_i bases) corresponding to the i th eigenvalue. The SVD in Eq. (2) also provides a basis U_i^* for left eigenvectors. The superscript $*$ means the complex conjugate. Furthermore, the SVD provides the bases for the right and left complimentary spaces that are required in the sequel.

Computation of Eigenvalue Sensitivity

To compute eigenvector derivatives (needed to compute eigenvector derivatives), let us take the derivative of the eigenvalue problem

$$A_i x_i^{(k)} = 0 \quad (5)$$

to obtain

$$\dot{A}_i^{(k)} x_i^{(k)} = -A_i \dot{x}_i^{(k)} = -\tilde{U}_i \Sigma_i \tilde{V}_i^H \dot{x}_i^{(k)} \quad (6)$$

where

$$\dot{A}_i^{(k)} = \dot{A} - \dot{\lambda}_i^{(k)} I \quad (7)$$

Premultiplying Eq. (6) by the hermitian transpose of left singular matrix, with the aid of Eq. (3) produces

$$\begin{bmatrix} \tilde{U}_i^H \\ U_i^H \end{bmatrix} \dot{A}_i^{(k)} x_i^{(k)} = \begin{bmatrix} \Sigma_i \tilde{V}_i^H \\ O \end{bmatrix} \dot{x}_i^{(k)} \quad (8)$$

Using the expansion of Eq. (4), the last V_i equations in Eq. (8) can be rewritten as

$$[U_i^H \dot{A} V_i] \alpha_i^{(k)} = \dot{\lambda}_i^{(k)} U_i^H V_i \alpha_i^{(k)} \quad (9)$$

The original form of Eq. (9) has been derived in Ref. 5. In Eq. (9), $\dot{\lambda}_i^{(k)}$ and $\alpha_i^{(k)}$ are the unknowns, and hence a subeigenvalue problem of order ν_i is defined. The eigenvalue derivative $\dot{\lambda}_i^{(k)}$ is associated with a specific eigenvector.

For the case of *distinct eigenvalue derivatives*, there exists a unique set of ν_i linearly independent vectors $\{\alpha_i^{(1)}, \dots, \alpha_i^{(\nu_i)}\}$ that are determined to a scaling factor, depending on the normalization of corresponding eigenvectors in Eq. (4). Henceforth, the case of repeated eigenvalues with repeated first eigenvalue derivatives is excluded. It is important to note that Eq. (9) always produces ν_i eigenvalue derivatives corresponding to the ν_i repeated eigenvalues, independent of the choice of eigenvectors. For simple eigenvalues, i.e., $\nu_i = 1$, Eq. (9) reduces to the familiar scalar equation¹

$$\dot{\lambda}_i = U_i^H \dot{A} V_i \quad (10)$$

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and $\alpha_i^{(1)}$ is simply the scaling factor for the i th eigenvector.

The corresponding left eigenvectors for the i th multiple eigenvalue can be expressed as

$$Y_i = U_i^* [\beta_i^{(1)}, \dots, \beta_i^{(v_i)}] \quad (11)$$

With the right eigenvectors of the i th multiple eigenvalue computed and normalized, the corresponding left eigenvectors can be chosen to satisfy the biorthogonality condition

$$Y_i^T X_i = I_{v_i} \quad (12)$$

if the expansion coefficients of Eq. (11) are computed as

$$\beta_i^T = (\alpha_i)^{-1} (U_i^H V_i)^{-1} \quad (13)$$

where $\alpha_i \triangleq [\alpha_i^{(1)}, \dots, \alpha_i^{(v_i)}]$ and $\beta_i \triangleq [\beta_i^{(1)}, \dots, \beta_i^{(v_i)}]$. Note that $\beta_i^{(k)}$ is simply the left eigenvector for the subeigenvalue problem

$$\beta_i^{(k)T} [U_i^H \dot{A} V_i] = \dot{\lambda}_i^{(k)} \beta_i^{(k)T} U_i^H V_i$$

Computation of Eigenvector Derivatives

Let the eigenvector derivative be expanded using the orthonormal right singular vectors

$$\dot{x}_i^{(k)} = \tilde{V}_i \tilde{\gamma}_i^{(k)} + V_{iyi}^{(k)}; \quad q = 1, \dots, v_i \quad (14)$$

The two terms in Eq. (14) represent the out-of-plane and in-plane components, respectively. Clearly, the above terminology is motivated by the fact that V_i spans the eigenvector space corresponding to the i th multiple eigenvalue. The above expression requires the calculation of the $(r_i \times 1)$ vector $\tilde{\gamma}_i^{(k)}$ and the $(v_i \times 1)$ vector $\gamma_i^{(k)}$.

First, consider the computation of the out-of-plane component, $\tilde{\gamma}_i^{(k)}$. For the special case when the multiplicity v_i , equals the order of the system n , $r_i = 0$ and $\tilde{\gamma}_i^{(k)}$ does not exist. If $r_i \geq 1$, the following derivations to compute $\tilde{\gamma}_i^{(k)}$ apply. Substituting Eqs. (4) and (14) into Eq. (8) gives

$$\tilde{U}_i^H \dot{A}_i^{(k)} V_i \alpha_i^{(k)} = -\tilde{U}_i \Sigma_i \tilde{\gamma}_i^{(k)} \quad (15)$$

Equation (15) represents r_i linearly independent equations and thus produces

$$\tilde{\gamma}_i^{(k)} = -\Sigma_i^{-1} \tilde{U}_i^H \dot{A}_i^{(k)} V_i \alpha_i^{(k)} \quad (16)$$

Note that, after the eigenvector $x_i^{(k)}$ is normalized, the out-of-plane component of the eigenvector derivative is completely determined.

To compute the in-plane component $\gamma_i^{(k)}$, consider the second derivative of the eigenvalue [Eq. (5)]

$$A_i^{(k)} x_i^{(k)} + 2\dot{A}_i^{(k)} x_i^{(k)} = -A_i x_i^{(k)} = -\tilde{U}_i \Sigma_i \tilde{V}_i^H x_i^{(k)} \quad (17)$$

Premultiplying Eq. (17) by the Hermitian transpose of left singular matrix and using Eqs. (4), (14), and (16), result in

$$[U_i^H \dot{A}_i^{(k)} V_i] \gamma_i^{(k)} = -\frac{1}{2} U_i^H [\ddot{A}_i^{(k)} - 2\dot{A}_i^{(k)} \tilde{V}_i \Sigma_i^{-1} \tilde{U}_i^H \dot{A}_i^{(k)}] V_i \alpha_i^{(k)} \quad (18)$$

From the assumption of distinct eigenvalue derivatives, Eq. (18) represents v_i linear equations that are underdetermined by exactly one rank. Clearly, the unique value of $\gamma_i^{(k)}$ is determined from the requirement for the consistent normalization

of eigenvectors, i.e.,

$$x_i^{(k)T} \dot{x}_i^{(k)} = \alpha_i^{(k)T} V_i^T \tilde{V}_i \gamma_i^{(k)} + \alpha_i^{(k)T} V_i^T V_i \gamma_i^{(k)} = 0 \quad (19)$$

For nonrepeated eigenvalues, the in-plane component becomes a scaling factor which must satisfy Eq. (19) only; Eq. (18) is not needed.

The second derivative of eigenvalue is required in Eq. (18). To compute the second derivative, premultiply Eq. (17) by the corresponding left eigenvector $y_i^{(k)T}$:

$$[y_i^{(k)T} [\ddot{A} - \ddot{\lambda}_i^{(k)} I] x_i^{(k)} + 2y_i^{(k)T} \dot{A}_i^{(k)} [\tilde{V}_i \gamma_i^{(k)} + V_i \gamma_i^{(k)}] = 0 \quad (20)$$

Now take the derivative of the k th multiple left eigenvalue problem and post multiply the result by the q th right multiple eigenvector

$$y_i^{(k)T} A_i x_i^{(q)} = -y_i^{(k)T} \dot{A}_i^{(k)} x_i^{(q)} = 0; \quad k, q = 1, \dots, v_i \quad (21)$$

which gives the vector equation, with the aid of Eq. (4)

$$y_i^{(k)T} \dot{A}_i^{(k)} V_i \alpha_i = 0 \quad (22)$$

where α_i is defined in Eq. (13). Since α_i is nonsingular, Eq. (22) implies that

$$y_i^{(k)T} \dot{A}_i^{(k)} V_i = 0 \quad (23)$$

Using Eq. (23), Eq. (20) yields

$$\ddot{\lambda}_i^{(k)} = y_i^{(k)T} \ddot{A} V_i \alpha_i^{(k)} + 2y_i^{(k)T} \dot{A}_i^{(k)} \tilde{V}_i \gamma_i^{(k)} \quad (24)$$

Concluding Remarks

The choice of bases obtained via singular value decomposition is clearly nonunique. All previous equations still hold for any set of bases spanning the eigenspaces corresponding to the repeated eigenvalue, provided the complementary bases are chosen to be orthogonal to the eigenspaces. However, if a complete set of bases corresponding to a multiple eigenvalue or even its order of multiplicity is unavailable or uncertain, the singular value decomposition procedure outlined in this report is strongly recommended for computing the four bases required for eigenvector derivatives for the multiple eigenvalue. As mentioned in Klema and Laub,⁴ "... since SVD is the only generally reliable way of calculating rank it follows that it is the only generally reliable way of calculating bases for these subspaces." Further work is needed to extend the computational algorithm to include the case of repeated eigenvalue derivatives.

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